

# DIFFERENTIAL POSETS HAVE STRICT RANK GROWTH: A CONJECTURE OF STANLEY

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ABSTRACT. We establish strict growth for the rank function of an  $r$ -differential poset. We do so by exploiting the representation theoretic techniques developed by Reiner and the author [1] for studying related Smith forms.

## 1. INTRODUCTION

For a positive integer  $r$ , an  $r$ -differential poset is a graded poset  $P$  with a minimum element, having all intervals and all rank cardinalities finite, satisfying

- (D1) If an element of  $P$  covers  $m$  others, then it will be covered by  $m + r$  others.
- (D2) If two elements of  $P$  have exactly  $m$  elements that they both cover, then there will be exactly  $m$  elements that cover them both.

We write  $P_n$  for the  $n$ th rank of  $P$  and set  $p_n := |P_n|$ . Considering the free  $\mathbb{Z}$ -module  $\mathbb{Z}P_n \cong \mathbb{Z}^{p_n}$  generated by the elements of  $P_n$ , define the *up* and *down* maps

$$\begin{aligned} U_n (= D_{n+1}^t) : \mathbb{Z}P_n &\rightarrow \mathbb{Z}P_{n+1} \\ D_n (= U_{n-1}^t) : \mathbb{Z}P_n &\rightarrow \mathbb{Z}P_{n-1} \end{aligned}$$

in which a basis element is sent by  $U_n$  (resp.  $D_n$ ) to the sum of all elements that cover (resp. are covered by) it. We shall often omit subscripts when the domain is clear. Setting

$$UD_n := U_{n-1}D_n \quad \text{and} \quad DU_n := D_{n+1}U_n,$$

conditions (D1,D2) can be rephrased as saying that

$$DU_n - UD_n = rI \quad \text{for each } n \geq 1.$$

The following result of Stanley concerning the spectra of  $DU_n$  is central to the theory and plays an important role in our analysis.

**Theorem 1.1** (Stanley [2]). *Let  $P$  be an  $r$ -differential poset. Then*

$$\det(DU_n + tI) = \prod_{i=0}^n (t + r(i+1))^{\Delta p_{n-i}},$$

where  $\Delta p_n := p_n - p_{n-1}$ .

Consequently, the rank sizes of a differential poset  $P$  weakly increase, that is  $p_0 \leq p_1 \leq p_2 \leq \dots$ . Though the growth of the rank function was recently studied in [3], an answer to the basic initial question of Stanley asking whether the rank sizes *strictly* increase has remained elusive; see [1, 2, 3]. The purpose of this paper is to resolve this question by establishing

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**Conjecture 1.2** (Stanley [2]). *An  $r$ -differential poset  $P$  has  $p_1 < p_2 < p_3 < \cdots$ .*

The techniques of this paper were inspired by those developed by Reiner and the author in [1] for towers of group algebras, to which we refer the reader for additional background, notation, and motivation. The key observation being that one can mimic enough of the character theory used in [1] with just two distinct elements  $t, s \in P_n$  having  $D^n t = D^n s = \hat{0}$ , the bottom element, for  $n \geq 2$ .

## 2. THE PROOF

We start with the following fundamental observation.

**Proposition 2.1** (Miller-Reiner [1]). *Let  $P$  be an  $r$ -differential poset, and  $j_{m-1} \triangleleft j_m$  any covering pair in  $P$  with the property that  $j_{m-1}, j_m$  both cover at most one element of  $P$ .*

*Then for any integer  $n \geq m$  one can extend this to a saturated chain*

$$j_{m-1} \triangleleft j_m \triangleleft j_{m+1} \triangleleft \cdots \triangleleft j_{n-1} \triangleleft j_n$$

*in which each  $j_\ell$  covers at most one element of  $P$ .*

Noting that  $r$  elements cover the bottom element of an  $r$ -differential poset and that all 1-differential posets are isomorphic to Young's lattice  $\mathbf{Y}$  up to rank  $n = 2$  (see Figure 1), we have the following

**Corollary 2.2.** *Let  $P$  be an  $r$ -differential poset. Then there exists a pair of chains*

$$t_0 \triangleleft t_1 \triangleleft t_2 \triangleleft \cdots \triangleleft t_n \triangleleft \cdots \quad \text{and} \quad s_0 \triangleleft s_1 \triangleleft s_2 \triangleleft \cdots \triangleleft s_n \triangleleft \cdots$$

*with the following properties:*

- (i)  $\text{rank}(t_n) = \text{rank}(s_n) = n$ ;
- (ii) if  $r = 1$  then  $t_n \neq s_n$  for  $n \geq 2$ , while  $t_0 = s_0$  and  $t_1 = s_1$ ;
- (iii) if  $r > 1$  then  $t_n \neq s_n$  for  $n \geq 1$ , while  $t_0 = s_0$ ;
- (iv) each  $t_n$  and  $s_n$  covers at most one element of  $P$ .

Fix one such pair of chains for each differential poset  $P$ , and refer to each using the notation of Corollary 2.2; see Figure 1. Further, when considering the matrix of an operator  $A \in \text{End}_{\mathbb{Z}}(\mathbb{Z}P_n)$ , it is understood that the *standard basis* consisting of the elements of  $P_n$  is to be considered, and ordered so that  $t_n$  indexes the first row and column of the matrix. Lastly, we remark that the notation is motivated by Young's lattice, viewed as the Bratteli diagram associated to the tower  $\{\mathbb{C}\mathfrak{S}_n\}_{n \geq 0}$ , where  $t_n$  and  $s_n$  correspond to the two linear representations of  $\mathfrak{S}_n$  for  $n \geq 2$ .

For nonnegative integers  $r, k, \ell$ , define

$$\ell!_{r,k} := (r \cdot \ell + k)(r \cdot (\ell - 1) + k) \cdots (r \cdot 1 + k)$$

with  $0!_{r,k} := 1$ .

**Proposition 2.3.** *Let  $P$  be an  $r$ -differential poset,  $k$  be a positive integer, and*

$$(2.1) \quad v_{n,k} := \sum_{j=0}^n (-1)^j \frac{U^j t_{n-j}}{(j+1)!_{r,k}}.$$

*Then  $(DU_n + kI)v_{n,k} = t_n$ .*

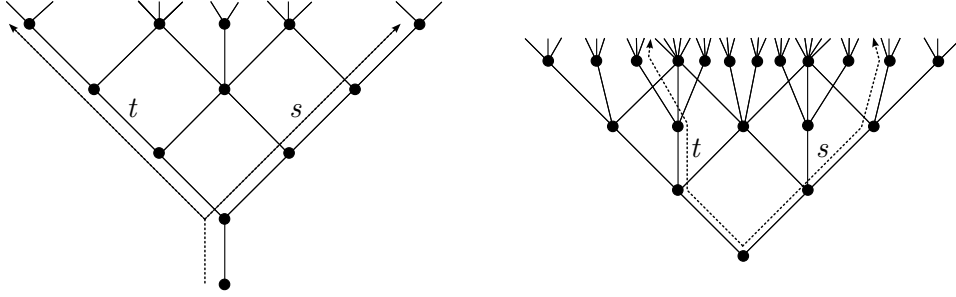


FIGURE 1. An illustration of Corollary 2.2 for a 1-differential poset (left) and a 2-differential poset (right).

*Proof.* For  $n = 0$ , we indeed have  $(DU_0 + kI) \frac{t_0}{r+k} = t_0$ . Inducting on  $n$ ,

$$\begin{aligned}
 (DU_{n+1} + kI)v_{n+1,k} &= (UD + (r+k)I) \left( \frac{t_{n+1}}{r+k} + \sum_{j=1}^{n+1} (-1)^j \frac{U^j t_{n+1-j}}{(j+1)!_{r,k}} \right) \\
 &= (UD + (r+k)I) \left( \frac{t_{n+1}}{r+k} - \frac{U}{r+k} \sum_{\ell=0}^n (-1)^\ell \frac{U^\ell t_{n-\ell}}{(\ell+1)!_{r,r+k}} \right) \\
 &= \frac{1}{r+k} ((UD + (r+k)I)t_{n+1} - U(DU + (r+k)I)v_{n,r+k}) \\
 &= \frac{1}{r+k} ((UD + (r+k)I)t_{n+1} - Ut_n) \\
 &= \frac{1}{r+k} (Ut_n + (r+k)t_{n+1} - Ut_n) \\
 &= t_{n+1},
 \end{aligned}$$

where the fourth equality is by induction, and the fifth follows from Corollary 2.2.  $\square$

**Corollary 2.4.** *Let  $P$  be an  $r$ -differential poset, and let  $k$  be a positive integer. Then  $DU_n + kI$  is invertible and the column vector of  $v_{n,k}$  forms the first column of the inverse  $(DU_n + kI)^{-1}$ .*

*Proof.* The first claim follows from Theorem 1.1, and the second follows from Proposition 2.3 and our convention of ordering the basis so that  $t_n$  indexes the first row and column of  $DU_n + kI$ .  $\square$

An integral matrix  $D = (d_{ij}) \in \text{Mat}_{n \times m}(\mathbb{Z})$  is said to be *in Smith form* if it is diagonal in the sense that  $d_{ij} = 0$  for  $i \neq j$ , and its diagonal entries are nonnegative and satisfy  $d_{11} | d_{22} | \cdots | d_{\min\{n,m\}}$ , in which case we set  $d_i := d_{ii}$  for each  $i$ . Recall that every integral matrix  $A \in \text{Mat}_{n \times m}(\mathbb{Z})$  can be brought into Smith form by an appropriate change of basis in  $\mathbb{Z}^n$  and  $\mathbb{Z}^m$ , i.e. there exist matrices  $P \in \text{GL}_n(\mathbb{Z})$  and  $Q \in \text{GL}_m(\mathbb{Z})$  for which  $PAQ = D$  is in Smith form. And though the matrices  $P, Q$  are not necessarily unique, the resulting Smith form  $D$  is, and its entries  $d_1, \dots, d_{\min\{n,m\}}$  are called the *Smith entries* of  $A$ , with  $d_{\min\{n,m\}}$  referred to as the *last Smith entry*. When  $A$  is square and invertible, we have the following well-known characterization of this last entry.

**Proposition 2.5** (cf. [1]). *Let  $A$  be an  $n \times n$  invertible (over  $\mathbb{Q}$ ) integral matrix, and let  $s$  be the smallest positive integer for which  $s \cdot A^{-1}$  is integral. Then  $d_n = s$ .*

**Theorem 2.6.** *Let  $P$  be an  $r$ -differential poset, let  $k$  and  $n$  be positive integers, and let  $d_{p_n}$  denote the last Smith entry of  $DU_n + kI$ . Then the following hold:*

- (i)  $(n+1)!_{r,k}$  divides  $d_{p_n}$  if  $r \geq 2$ ;
- (ii)  $(n-1)!_{r,k} \cdot (n+1+k)$  divides  $d_{p_n}$  if  $r = 1$ .

*Proof.* By Proposition 2.5,  $d_{p_n}$  is the smallest positive integer for which

$$d_{p_n} \cdot (DU_n + kI)^{-1} \in \text{Mat}_{p_n \times p_n}(\mathbb{Z}).$$

It thus suffices to show that the claimed divisor  $d$  of  $d_{p_n}$  is the smallest positive integer  $s$  for which the first column of  $s \cdot (DU_n + kI)^{-1}$  is integral, or equivalently  $s \cdot v_{n,k} \in \mathbb{Z}P_n$  by Corollary 2.4.

For  $r \geq 2$  it is clear that  $d \cdot v_{n,k} \in \mathbb{Z}P_n$  by (2.1). Moreover, by Corollary 2.2

$$\langle U^j t_{n-j}, s_n \rangle = \langle t_{n-j}, D^j s_n \rangle = \langle t_{n-j}, s_{n-j} \rangle = \begin{cases} 1 & \text{if } j = n \\ 0 & \text{if } j \leq n-1, \end{cases}$$

where  $\langle -, - \rangle$  denotes the bilinear form obtained by decreeing that  $\langle x, y \rangle = \delta_{x,y}$  for  $x, y \in P_n$ . It follows that  $d$  is the smallest positive integer for which  $d \cdot v_{n,k} \in \mathbb{Z}P_n$ .

Similarly, for  $r = 1$  we have that

$$(2.2) \quad \langle U^j t_{n-j}, s_n \rangle = \begin{cases} 1 & \text{if } j = n \text{ or } n-1 \\ 0 & \text{if } j \leq n-2. \end{cases}$$

Considering the  $j = n-1$  and  $j = n$  terms of  $v_{n,k}$ , we have

$$\frac{U^{n-1} t_1}{n!_{1,k}} - \frac{U^n t_0}{(n+1)!_{1,k}} = \frac{U^n t_0}{n!_{1,k}} - \frac{U^n t_0}{(n+1)!_{1,k}} = \frac{U^n t_0}{(n-1)!_{1,k} \cdot (n+1+k)},$$

and thus

$$(2.3) \quad v_{n,k} = \sum_{j=0}^{n-2} (-1)^j \frac{U^j t_{n-j}}{(j+1)!_{1,k}} + (-1)^{n-1} \frac{U^n t_0}{(n-1)!_{1,k} \cdot (n+1+k)}.$$

It follows that indeed  $d \cdot v_{n,k} \in \mathbb{Z}P_n$ . Further, (2.2) and (2.3) together imply that  $d$  is the smallest such integer.  $\square$

**Corollary 2.7.** *Let  $P$  be an  $r$ -differential poset. Then  $p_1 < p_2 < p_3 < \dots$ .*

*Proof.* The following argument was given by Reiner and the author in [1], verbatim, but in the context of differential posets associated with towers of group algebras; see [1, Proof of Cor. 6.13].

For a fixed  $n \geq 2$ , to show that  $\Delta p_n > 0$ , apply Theorem 2.6 with the positive integer  $k$  chosen so that  $r+k$  is a prime that divides *none* of

$$2r+k, 3r+k, \dots, (n+1)r+k$$

(e.g. pick  $p$  to be a prime larger than  $(n+1)r$  and take  $k = p - r$ ). Theorem 2.6 tells us that the prime  $r+k$  divides the last Smith form entry  $d_{p_n}$  for  $DU_n + kI$  over  $\mathbb{Z}$ , and hence it also divides

$$\det(DU_n + kI) = (r+k)^{\Delta p_n} (2r+k)^{\Delta p_{n-1}} \dots (nr+k)^{\Delta p_1} ((n+1)r+k)^{p_0}$$

according to Theorem 1.1. Since  $r+k$  is a prime that can only divide the first factor on the right, it must be that  $\Delta p_n > 0$ .  $\square$

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